Nonlinear couplings and tree amplitudes in gauge theories

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Following a remark advanced by Feynman, we study the connection between the form of the nonlinear vertices involving gauge particles and the Abelian gauge invariance of physical tree amplitudes. We show that this requirement, together with some natural assumptions, fixes uniquely the structure of the Yang-Mills theory. However, the constraints imposed by the above property are not sufficient to single out the gauge theory of gravitation.

I. INTRODUCTION

In the Yang-Mills theory, the source of the Yang-Mills fields is the conserved color current. Since these fields carry color, these will self-interact leading to a non-Abelian gauge theory [1]. Similarly, the source of the gravitational fields is the energy-momentum tensor, a quantity which is locally conserved. These fields carry energy and momentum and hence must couple to themselves. The non-Abelian gauge theory of gravitation, which is invariant under local gauge transformations, is identical to Einstein's theory [2]. There has been much fundamental work on basic aspects of the non-Abelian gauge theories [3,4,5,6,7,8,9]. Feynman [3] has shown that in these theories, the tree amplitudes involving free external gauge fields must be invariant under Abelian gauge transformations of the external fields. He remarked that this property may be used in order to investigate, in an alternative way, the structure of the nonlinear graviton interactions.

The purpose of this work is to study the question whether the above property of physical

tree amplitudes is sufficient to determine completely the form of the nonlinear interactions between the gauge particles. We consider this problem in section II, first in the simpler context of the Yang-Mills theory. We assume that the nonlinear interactions between the gluons are local and involve only dimensionless coupling constants. We find that in this case the answer to the above question is affirmative, basically due to the absence of gluon vertices of higher degree than four. In section III, we work out the corresponding expressions for gravity, whose algebraic complexity is much greater. We assume that the interactions between the gravitons are local and involve only two derivatives of these fields. This allows for the presence of graviton self-couplings to all orders. In the gravity case, it is always possible to make a local redefinition of the basic fields, such that the physical amplitudes will be the same [9]. We argue that, even accounting for this possibility, the Abelian gauge invariance of the tree amplitudes does not yield enough constraints to fix the form of the nonlinear graviton couplings.

We report for simplicity only the results for pure gauge theories, since the problem we study is basically connected with the self-interaction of gauge particles. We have verified that the introduction of matter fields adds only a further algebraic complication, without modifying the above conclusions. Finally, we mention that other interesting aspects of tree amplitudes in gauge theories have been discussed recently in the literature [10,11,12].

II. THE YANG-MILLS THEORY

We start with the Yang-Mills case, characterized by a gauge field A^a_α, where a denotes the color index and α is a Lorentz index. The quadratic part of the Yang-Mills Lagrangian

$$\mathcal{L}^2_{YM}\left(A\right) = \frac{1}{4}\left(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta\right)\left(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta\right), \tag{2.1}$$

is invariant under the Abelian gauge transformation

$$A^a_\alpha \rightarrow A^a_\alpha + \partial_\alpha \omega^a. \tag{2.2}$$

This leads in momentum space to the free equation of motion

$$\left(\eta_{\alpha\beta}k^2 - k_\alpha k_\beta\right) A^a_\beta \left(k\right) = 0, \tag{2.3}$$

which is invariant under the gauge transformation

$$\delta A^a_\alpha \left(k\right) = \omega^a k_\alpha. \tag{2.4}$$

We now consider the interactions between the gluons, which we assume to be local and characterized by dimensionless coupling constants. This natural assumption allows for vertices involving 3 gluons with one derivative term and 4 gluons with no derivatives, but precludes the presence of higher order gluon self-couplings. In this case, using Bose symmetry and Lorentz invariance and disregarding total derivatives terms, we can write the interaction Lagrangian as follows

$$\begin{aligned}\mathcal{L}^I_{YM}\left(A\right) = &\left(g\, f_{abc} + e_0\, d_{abc}\right)\left(\partial_\nu A^a_\mu\right) A^b_\mu A^c_\nu + \\ &\left(l_0\, f_{abe} f_{cde} + l_1\, d_{abe} d_{cde} + l_2\, \delta_{ab}\delta_{cd}\right) A^a_\mu A^b_\nu A^c_\mu A^d_\nu + \\ &\left(l_3\, d_{abe} d_{cde} + l_4\, \delta_{ab}\delta_{cd}\right) A^a_\mu A^b_\mu A^c_\nu A^d_\nu.\end{aligned} \tag{2.5}$$

Here f_{abc} denote the antisymmetric color structure constants of the gauge group SU(N) and d_{abc} are the symmetric color factors. The coupling constant g sets the scale of the gluon interactions and e_0, l_i are dimensionless couplings which must be determined.

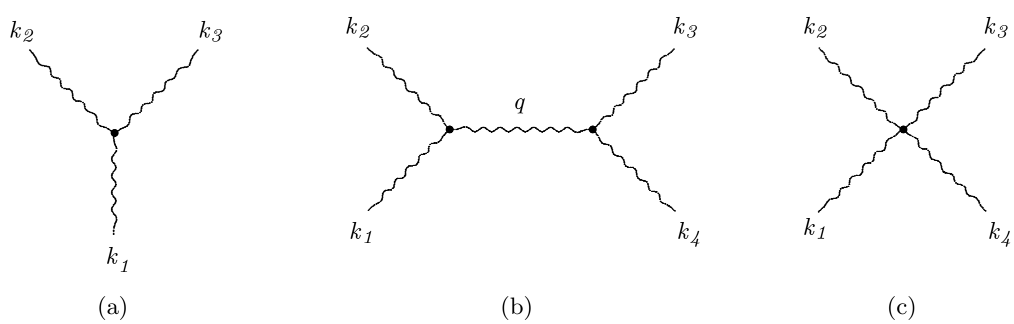

FIG. 1. Basic tree diagrams involving gauge particles. All momenta are inwards with $\sum k_i = 0$.

We proceed by imposing the condition that the gluon tree amplitudes should be invariant under the Abelian gauge transformation given by (2.4). This property [3] follows in consequence of the fact that the external lines satisfy the free equation of motion (2.3). We

use this constraint on the 3-gluon vertex shown in Fig. 1a and perform a gauge transformation on the field $A^a_\alpha(k_1)$. Since the trilinear gluon coupling proportional to $g\, f_{abc}$ satisfies identically the above constraint, when we make use of momentum conservation, we find the condition that

$$e_0 \left(k_{2\beta}\; k_{3\gamma} - k_2 \cdot k_3 \eta_{\beta\gamma}\right) \omega^a\, d_{abc} A^b_\beta(k_2) A^c_\gamma(k_3) = 0. \tag{2.6}$$

Because k_2 and k_3 are arbitrary and independent momenta, this equation requires the vanishing of the coupling constant e_0

$$e_0 = 0. \tag{2.7}$$

Therefore, in this case the Abelian gauge invariance determines basically the structure of the trilinear vertex. As we shall see, this special feature does not occur in the gravity case, which is much more complicated algebraically.

We now evaluate the contributions from the graph in Fig. 1b and its permutations to the gluon-gluon scattering amplitude. In order to perform these calculations, it is simpler to use the Feynman propagator $\eta_{\mu\nu}/q^2$. In view of the Abelian gauge invariance of this amplitude, we must equate the negative of the gauge variation of these contributions to the corresponding variations associated with the 4-gluon vertex shown in Fig 1c. Then, under a gauge transformation of the gluon field $A^a_\alpha(k_1)$, we find that

$$\begin{aligned}[\delta\; tree]_{1c} = -g^2 \Big[& f_{abe} f_{cde} \left(k_{1\beta}\eta_{\sigma\gamma} + k_{1\sigma}\eta_{\beta\gamma} - 2k_{1\gamma}\eta_{\beta\sigma}\right) + \\ & f_{ace} f_{bde} \left(k_{1\gamma}\eta_{\sigma\beta} + k_{1\sigma}\eta_{\beta\gamma} - 2k_{1\beta}\eta_{\gamma\sigma}\right)\Big] \\ & \times \omega^a A^b_\beta(k_2) A^c_\gamma(k_3) A^d_\sigma(k_4), \end{aligned} \tag{2.8}$$

where we have used the Jacobi identity

$$f_{abe} f_{cde} + f_{ace} f_{dbe} + f_{ade} f_{bce} = 0 \tag{2.9}$$

to eliminate contributions proportional to $f_{ade} f_{bce}$.

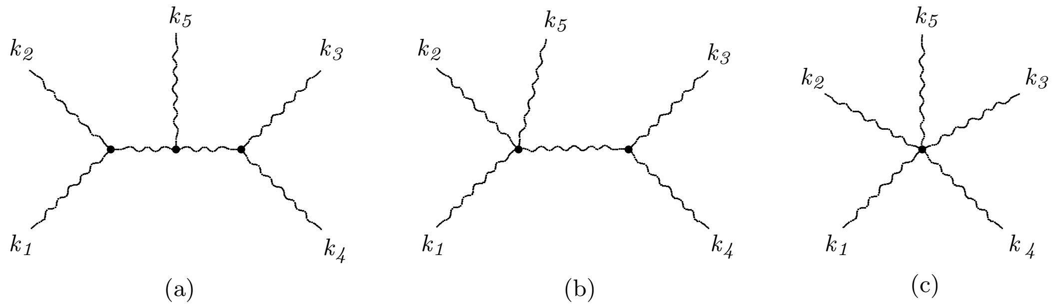

FIG. 2. Higher order tree amplitude containing nonlinear couplings of gauge particles

We can now express the gauge variation on the left hand side of (2.8) in terms of the parameters introduced in (2.5). Using relations like

$$f_{abe}f_{cde} = \frac{2}{N}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) + d_{ace}d_{dbe} - d_{ade}d_{bce}, \tag{2.10}$$

and identifying the coefficients of the independent structures appearing in (2.8), we obtain the following relations:

$$\begin{aligned} l_1 &= -l_3 = l_0 - \frac{g^2}{4}, \\ l_2 &= -l_4 = \frac{2}{N}\left(l_0 - \frac{g^2}{4}\right). \end{aligned} \tag{2.11}$$

We thus see that the parameters l_i have not been fully determined by the gauge invariance property of the gluon-gluon scattering amplitude. However, we can now apply this condition also to the 5-gluon tree amplitude represented by diagrams like the one shown in figures 2a and 2b. Due to the absence of direct 5-gluon couplings, and using the equations (2.11), it is straightforward to show that this constraint yields a further relation:

$$l_0 = \frac{g^2}{4}. \tag{2.12}$$

Together with (2.11), this relation implies the vanishing of the coupling constants l_i ($i = 1, 2, 3, 4$). Substituting these results in equation (2.5), and using (2.1) and (2.7), we arrive at the well known expression for the Yang-Mills Lagrangian

$$\mathcal{L}_{YM}(A) = \tfrac{1}{4}\Big(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta + g\, f_{abc} A^b_\alpha A^c_\beta\Big)$$
$$\Big(\partial_\beta A^a_\alpha - \partial_\alpha A^a_\beta + g\, f_{ab'c'} A^{b'}_\alpha A^{c'}_\beta\Big)\,. \tag{2.13}$$

III. THE GRAVITATIONAL FIELD

In this case, it is convenient to introduce a symmetric tensor field $h_{\mu\nu}$ representing the deviation of the metric tensor $g_{\mu\nu}$ from the flat space Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}, \tag{3.1}$$

where κ is the usual gravitational constant. Gauge symmetry and Lorentz invariance enable us to get the linearized gravitational Lagrangian

$$\mathcal{L}^2(h) = \tfrac{1}{2} h_{\mu\nu,\alpha} h_{\mu\nu,\alpha} - \frac{1}{2} h_{\mu\mu,\alpha} h_{\nu\nu,\alpha} +$$
$$h_{\mu\mu,\alpha} h_{\alpha\nu,\nu} - h_{\mu\nu,\nu} h_{\mu\alpha,\alpha} \quad , \tag{3.2}$$

where the index after a comma indicates differentiation. Although we are not making explicit the distinction between up and down indices, the Minkowski metric tensor $\eta_{\mu\nu}$ is implicitly present in all the contractions of pairs of identical indices (e. g. $h_{\mu\mu} = \eta_{\mu\nu} h_{\mu\nu}$).

It is easy to verify that the above Lagrangian is invariant under the Abelian gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}. \tag{3.3}$$

By varying this Lagrangian one obtains in momentum space the equation of motion satisfied by a free graviton

$$\Big(k^2 \eta_{\alpha\mu}\eta_{\beta\nu} - k_\mu k_\alpha \eta_{\beta\nu} - k_\mu k_\beta \eta_{\alpha\nu} + k_\alpha k_\beta \eta_{\mu\nu}\Big)\, h_{\mu\nu}(k) = 0, \tag{3.4}$$

which is invariant under the gauge transformation

$$\delta h_{\mu\nu}(k) = k_\nu \xi_\mu + k_\mu \xi_\nu \tag{3.5}$$

In order to proceed, we need to parametrize the general structure of the graviton self-interactions, which we assume to involve products of fields with two derivative indices. The algebraic complexity is now so great that we have made use of computer algebra to do the calculations. We start constructing the 3-graviton vertex $\mathcal{L}^3$ as a sum over all possible independent trilinear products of fields with two derivative terms. When we write all possible such products and use Lorentz invariance, we find an expression involving 16 independent constants a_i

$$
\begin{aligned}
\mathcal{L}^3(h) = \kappa\,(\; & a_1\, h_{\mu\nu}\, h_{\alpha\beta,\mu}\, h_{\nu\alpha,\beta} + a_2\, h_{\mu\nu}\, h_{\alpha\alpha,\mu}\, h_{\nu\beta,\beta} + a_3\, h_{\mu\nu}\, h_{\mu\alpha,\alpha}\, h_{\nu\beta,\beta} + \\
& a_4\, h_{\mu\nu}\, h_{\mu\alpha,\nu}\, h_{\alpha\beta,\beta} + a_5\, h_{\mu\nu}\, h_{\alpha\beta,\mu}\, h_{\alpha\beta,\nu} + a_6\, h_{\mu\nu}\, h_{\mu\alpha,\beta}\, h_{\nu\alpha,\beta} + \\
& a_7\, h_{\mu\mu}\, h_{\nu\alpha,\beta}\, h_{\nu\alpha,\beta} + a_8\, h_{\mu\mu}\, h_{\nu\alpha,\nu}\, h_{\alpha\beta,\beta} + a_9\, h_{\mu\nu}\, h_{\mu\nu,\alpha}\, h_{\alpha\beta,\beta} + \\
& a_{10}\, h_{\mu\mu}\, h_{\nu\alpha,\nu}\, h_{\beta\beta,\alpha} + a_{11}\, h_{\mu\nu}\, h_{\alpha\alpha,\beta}\, h_{\mu\nu,\beta} + a_{12}\, h_{\mu\nu}\, h_{\alpha\alpha,\mu}\, h_{\beta\beta,\nu} + \\
& a_{13}\, h_{\mu\nu}\, h_{\mu\alpha,\nu}\, h_{\beta\beta,\alpha} + a_{14}\, h_{\mu\nu}\, h_{\nu\alpha,\beta}\, h_{\mu\beta,\alpha} + a_{15}\, h_{\mu\mu}\, h_{\nu\alpha,\beta}\, h_{\nu\beta,\alpha} + \\
& a_{16}\, h_{\mu\mu}\, h_{\nu\nu,\alpha}\, h_{\beta\beta,\alpha} \;) \qquad .
\end{aligned}
\tag{3.6}
$$

The next steps are done in correspondence with the ones in the Yang-Mills theory. We attempt to determine these constants, using the requirement of gauge invariance under the transformation (3.5) of the 3-graviton vertex associated with Fig. 1a. Using the equation of motion (3.4) for the free gravitons and momentum conservation, this results in a set of 7 independent equations for the 16 parameters, which yield the following relations:

$$
\begin{aligned}
a_1 &= a_{14} - 6\,a_8 - 6\,a_{15} - 8\,a_{16} - 8\,a_{10} - 4\,a_{11} - 4\,a_9 \\
a_2 &= -6\,a_8 - 8\,a_{15} - 8\,a_{16} - 8\,a_{10} + a_{13} - 2\,a_{11} - 2\,a_9 - 4\,a_{12} \\
a_3 &= 14\,a_{15} + 12\,a_8 + 16\,a_{16} + 16\,a_{10} - 2\,a_{13} + 4\,a_{11} + 4\,a_9 + 4\,a_{12} - a_{14} \\
a_4 &= -a_{14} + 2\,a_{15} - 2\,a_{13} \\
a_5 &= 3\,a_8 + 3\,a_{15} + 4\,a_{16} + 4\,a_{10} + 2\,a_{11} + 2\,a_9 \\
a_6 &= -2\,a_{15} + 2\,a_{13} \\
2\,a_7 &= -3\,a_8 - 3\,a_{15} - 4\,a_{16} - 4\,a_{10}.
\end{aligned}
\tag{3.7}
$$

We remark that after inserting (3.7) into (3.6) the resulting expression is such that the coefficient of a_{14} is a total derivative.

In contrast to the situation in the Yang-Mills theory [see eq. (2.7)] we see that in this case we do not have enough conditions to determine all the parameters of the trilinear graviton couplings. All we can do is to express $\mathcal{L}^3$ as a function of the parameters which appear on the right hand side of eq. (3.7), which we denote collectively be the set $\tilde{a} \equiv a_8, \cdots, a_{16}$.

It is appropriate to comment here on the possibility of making a local transformation of the fields so that

$$h'_{\mu\nu} = h_{\mu\nu} + \kappa \, (\, A_1 \eta_{\mu\nu} \, (h_{\alpha\alpha})^2 + A_2 \eta_{\mu\nu} h_{\alpha\beta} h_{\beta\alpha} + \\ A_3 h_{\mu\alpha} h_{\nu\alpha} + A_4 h_{\mu\nu} h_{\alpha\alpha} \,) + \cdots, \tag{3.8}$$

where $\cdots$ denote terms of higher order in κ. Note that the Abelian gauge transformation (3.5) is the same for both fields $h_{\mu\nu}$ and $h'_{\mu\nu}$. Since the terms of order κ in (3.8) involve 4 arbitrary parameters, it is possible to make a redefinition of the fields such that the number of independent parameters in (3.6) may be reduced from 9 to 5. Even allowing for this possibility, we see that in contrast to the Yang-Mills case, there remains a basic indetermination of the trilinear graviton couplings.

Following the analysis done in the Yang-Mills case, we may evaluate the contributions from the graph 1b and its permutations to the graviton-graviton tree amplitude, in terms of the parameters present in the set $\tilde{a}$. Since the gauge invariance condition of the physical tree amplitude should be valid for any gauge-fixing term added onto (3.2), it will be convenient to choose this so that the graviton propagator becomes [3]

$$P_{\mu\nu\alpha\beta}(q) = \frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}}{2q^2}. \tag{3.9}$$

(We have verified, in the case of the gravitational Compton scattering by scalar particles, that no additional information is obtained by considering a more general class of gauges.) The result of this evaluation, involving quadratic functions of the parameters $\tilde{a}_i$ which are excessively long to write down here, will be employed subsequently.

Next we must parametrize the structure of the 4-graviton vertex $\mathcal{L}^4$ indicated in Fig 1c, in terms of all possible quadrilinear products of fields with two derivatives indices. Proceeding in this way, we find for $\mathcal{L}^4$ the following expression involving 43 independent constants:

$$
\begin{aligned}
\mathcal{L}^4(h) = \kappa^2 \big(\quad & b_1\, h_{\mu\nu}\, h_{\mu\nu}\, h_{\alpha\beta,\rho}\, h_{\alpha\rho,\beta} + b_2\, h_{\mu\mu}\, h_{\nu\nu}\, h_{\alpha\beta,\rho}\, h_{\alpha\rho,\beta} + \\
b_3\, h_{\mu\nu}\, h_{\mu\nu}\, h_{\alpha\alpha,\beta}\, h_{\rho\rho,\beta} + {} & b_4\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\beta,\alpha}\, h_{\nu\rho,\rho} + b_5\, h_{\mu\nu}\, h_{\mu\nu}\, h_{\alpha\beta,\alpha}\, h_{\beta\rho,\rho} + \\
b_6\, h_{\mu\mu}\, h_{\nu\nu}\, h_{\alpha\beta,\alpha}\, h_{\beta\rho,\rho} + {} & b_7\, h_{\mu\mu}\, h_{\nu\nu}\, h_{\alpha\beta,\rho}\, h_{\alpha\beta,\rho} + b_8\, h_{\mu\mu}\, h_{\nu\nu}\, h_{\alpha\alpha,\beta}\, h_{\rho\rho,\beta} + \\
b_9\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\beta,\rho}\, h_{\alpha\nu,\rho} + {} & b_{10}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\mu,\nu}\, h_{\rho\rho,\beta} + b_{11}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\mu,\beta}\, h_{\nu\rho,\rho} + \\
b_{12}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\nu,\beta}\, h_{\alpha\rho,\rho} + {} & b_{13}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\mu,\nu}\, h_{\beta\rho,\rho} + b_{14}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\mu,\nu}\, h_{\rho\rho,\beta} + \\
b_{15}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\rho,\beta}\, h_{\alpha\nu,\rho} + {} & b_{16}\, h_{\mu\mu}\, h_{\alpha\nu}\, h_{\alpha\nu,\beta}\, h_{\beta\rho,\rho} + b_{17}\, h_{\mu\mu}\, h_{\alpha\nu}\, h_{\alpha\beta,\rho}\, h_{\nu\rho,\beta} + \\
b_{18}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\rho,\mu}\, h_{\nu\rho,\beta} + {} & b_{19}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\rho,\nu}\, h_{\beta\rho,\alpha} + b_{20}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\rho,\alpha}\, h_{\beta\nu,\rho} + \\
b_{21}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\rho,\mu}\, h_{\beta\nu,\rho} + {} & b_{22}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\mu\rho,\nu}\, h_{\alpha\rho,\beta} + b_{23}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\rho,\mu}\, h_{\beta\rho,\nu} + \\
b_{24}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\mu,\rho}\, h_{\beta\nu,\rho} + {} & b_{25}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\nu,\rho}\, h_{\alpha\beta,\rho} + b_{26}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\nu,\alpha}\, h_{\beta\rho,\rho} + \\
b_{27}\, h_{\mu\nu}\, h_{\mu\nu}\, h_{\alpha\beta,\rho}\, h_{\alpha\beta,\rho} + {} & b_{28}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\beta,\rho}\, h_{\mu\nu,\rho} + b_{29}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\beta,\mu}\, h_{\nu\rho,\rho} + \\
b_{30}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\mu,\nu}\, h_{\beta\rho,\rho} + {} & b_{31}\, h_{\mu\nu}\, h_{\mu\nu}\, h_{\alpha\beta,\alpha}\, h_{\rho\rho,\beta} + b_{32}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\beta,\nu}\, h_{\rho\rho,\alpha} + \\
b_{33}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\nu,\rho}\, h_{\alpha\rho,\beta} + {} & b_{34}\, h_{\mu\mu}\, h_{\alpha\nu}\, h_{\beta\beta,\nu}\, h_{\rho\rho,\alpha} + b_{35}\, h_{\mu\mu}\, h_{\nu\nu}\, h_{\alpha\alpha,\beta}\, h_{\beta\rho,\rho} + \\
b_{36}\, h_{\mu\mu}\, h_{\alpha\nu}\, h_{\beta\beta,\nu}\, h_{\alpha\rho,\rho} + {} & b_{37}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\rho,\mu}\, h_{\beta\rho,\nu} + b_{38}\, h_{\mu\nu}\, h_{\alpha\alpha}\, h_{\beta\mu,\rho}\, h_{\beta\nu,\rho} + \\
b_{39}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\mu\rho,\nu}\, h_{\alpha\beta,\rho} + {} & b_{40}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\rho,\mu}\, h_{\beta\nu,\rho} + b_{41}\, h_{\mu\mu}\, h_{\alpha\nu}\, h_{\alpha\nu,\beta}\, h_{\rho\rho,\beta} + \\
b_{42}\, h_{\mu\nu}\, h_{\alpha\beta}\, h_{\alpha\beta,\mu}\, h_{\rho\rho,\nu} + {} & b_{43}\, h_{\mu\nu}\, h_{\alpha\mu}\, h_{\beta\nu,\alpha}\, h_{\rho\rho,\beta} \big) \quad .
\end{aligned}
\tag{3.10}
$$

From the gauge invariance condition, one expects that a change in the gravitational field $\delta h_{\mu\nu}$ given by (3.5), should have no effect on the graviton-graviton tree amplitude. Imposing this requirement and using the results mentioned in the previous equations, we obtain a relation which can be written in correspondence with (2.8) as

$$
[\delta\ tree(b_i)]_{1c} = -\,[\delta\ tree(\tilde{a})]_{1b}\,, \tag{3.11}
$$

where $[\delta\ tree]_{1b}$ represents the gauge variation associated with the diagram in Fig. 1b and its corresponding permutations. It is expressed as a function of the independent coupling constants $\tilde{a}_i$ left over from the analysis of the trilinear graviton vertices. The left-hand side of eq. (3.11), denotes the gauge variation resulting from the contributions associated with the graph in Fig. 1c, which is a function of the independent constants $b_1 \cdots b_{43}$, which parametrize the 4-graviton vertex in (3.10).

We now gather together the terms with the same structure and set the coefficients of all independent structures in (3.11) equal to zero. We then obtain a system which comprises 27 algebraically independent equations, expressing certain linear combinations of the b_i in terms of quadratic functions of the parameters $\tilde{a}_i$. Clearly, this set of independent equations cannot determine all the parameters b_i, nor can it lead to any additional relations among the $\tilde{a}_i$. The solution of the above system in given by a set of equations where the 27 coefficients b_i ($i = 1, 2, \cdots, 26, 27$) are expressed in terms of the remaining 16 coefficients b_i ($i = 28, 29, \cdots, 42, 43$) and of the parameters $\tilde{a}_i$. We write here explicitly only a few typical equations:

$$
\begin{aligned}
8\,b_1 \;&=-\; 4\,b_{29} - 8\,b_{42} + 12\,a_{11}a_8 + a_9a_{13} + 16\,a_{16}a_{11} + 6\,a_9a_8 + 16\,a_{10}a_{11} + \\
&\qquad 14\,a_9a_{11} + 5\,a_9a_{15} + 8\,a_9a_{10} + 8\,a_9a_{16} + 12\,a_{15}a_{11} + 6\,{a_9}^2 + 8\,{a_{11}}^2 \\
8\,b_2 \;&=-\; 2\,b_{28} - 4\,b_{30} - 2\,b_{32} - 4\,b_{33} + 2\,b_{38} - 8\,b_{39} - 2\,b_{43} - 3\,a_9a_8 - \\
&\qquad 3\,a_{11}a_8 - 4\,a_{10}a_{11} - 16\,{a_{10}}^2 - 4\,a_9a_{10} - 24\,a_{10}a_8 + 4\,a_{13}a_{11} + \\
&\qquad 4\,a_9a_{13} + 8\,a_{13}a_8 - 18\,{a_{15}}^2 + 10\,a_{15}a_{13} - 35\,a_{15}a_{10} - 7\,a_9a_{15} - \\
&\qquad 7\,a_{15}a_{11} - 26\,a_{15}a_8 + 11\,a_{10}a_{13} - {a_{13}}^2 - 4\,a_{16}a_{11} + 12\,a_{16}a_{13} - \\
&\qquad 36\,a_{15}a_{16} - 32\,a_{16}a_{10} - 4\,a_9a_{16} - 24\,a_{16}a_8 - 9\,{a_8}^2 - 16\,{a_{16}}^2 \\
\vdots \;&=\; \vdots \\
2\,b_{26} \;&=\; 2\,b_{28} + 4\,b_{33} - 4\,b_{38} - 12\,a_9a_8 - 4\,{a_9}^2 - 4\,{a_{11}}^2 - 12\,a_{11}a_8 - \\
&\qquad 16\,a_{10}a_{11} - 8\,a_9a_{11} - 16\,{a_{10}}^2 - 16\,a_9a_{10} - 24\,a_{10}a_8 - 8\,a_{13}a_{11} - \\
&\qquad 8\,a_9a_{13} - 12\,a_{13}a_8 + 4\,{a_{15}}^2 - 14\,a_{15}a_{13} - 8\,a_{15}a_{10} - 4\,a_9a_{15} - \\
&\qquad 4\,a_{15}a_{11} - 6\,a_{15}a_8 + 16\,a_{10}a_{13} + {a_{13}}^2 - 16\,a_{16}a_{11} - 16\,a_{16}a_{13} - \\
&\qquad 8\,a_{15}a_{16} - 32\,a_{16}a_{10} - 16\,a_9a_{16} - 24\,a_{16}a_8 - 9\,{a_8}^2 - 16\,{a_{16}}^2 \\
16\,b_{27} \;&=\; 8\,b_{42} - 4\,a_9a_{16} - 4\,{a_9}^2 - 3\,a_9a_{15} - 4\,a_9a_{10} - 3\,a_9a_8 - 12\,a_9a_{11} - \\
&\qquad 12\,a_{11}a_8 - 16\,a_{16}a_{11} - 16\,a_{10}a_{11} - 12\,a_{15}a_{11} - 8\,{a_{11}}^2 .
\end{aligned}
\tag{3.12}
$$

Although much more complicated in detail, these relations are basically similar to the one encountered in the Yang-Mills case [see eq. (2.11)]. The crucial difference occurs when attempting, in parallel to the procedure used in the Yang-Mills case, to apply the gauge invariance condition to the 5-graviton tree amplitude. Now, there exists a basic 5-graviton

vertex, shown in Fig 2c, which must be parametrized in terms of the most general sum of independent products involving five graviton fields with two derivatives. This parametrization can be done in terms of a very large number of new constants, which we denote by the set c_i. Following closely the analysis after equation (3.11), is is clear that the Abelian gauge invariance condition will merely lead to some relations expressing certain c_i in terms of the remaining c_i and of the parameters $\tilde{a}_i$ and b_i left over from the previous analysis.

It is evident that this behavior is quite general, in view of the fact that the graviton self-couplings occur to all orders. We thus conclude that the constraint of Abelian gauge invariance of the physical tree amplitudes does not determine completely the from of the nonlinear graviton interactions. It is only when we impose the condition that the theory should be invariant under (infinitesimal) non-Abelian gauge transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \kappa \left(\xi^{\sigma}_{,\mu} h_{\sigma\nu} + \xi^{\sigma}_{,\nu} h_{\sigma\mu} + \xi^{\sigma} h_{\mu\nu,\sigma} \right) . \tag{3.13}$$

that it becomes essentially determined. For example, using the parametrization given by (3.6), we find in this case for the trilinear graviton vertex:

$$\begin{array}{llll}
a_1 = 1 + a_{14} \, , & 2\, a_2 = -3 - 2\, a_{15} \, , & a_3 = 1 - a_{14} \, , & a_4 = a_3 \, , \\
2\, a_5 = -1 \, , & a_6 = -1 \, , & 4\, a_7 = 1 \, , & 2\, a_8 = -1 - 2\, a_{15} \\
a_9 = -1 \, , & 2\, a_{10} = 1 \, , & a_{11} = 1 \, , & 2\, a_{12} = 1 \, , \\
2\, a_{13} = 2\, a_{15} - 1 \, , & 4\, a_{16} = -1 & . &
\end{array} \tag{3.14}$$

We remark that the structures which multiply a_{14} and a_{15} add up to total derivative terms. Since total derivatives are not relevant for our purpose, this result is equivalent to the one obtained from the Einstein's general relativity. Then, the theory becomes consistent with the existence of a locally conserved energy-momentum "tensor" of matter and gravitation [2].

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